

DISTRIBUTIONAL FRACTIONAL POWERS OF SIMILAR OPERATORS. APPLICATIONS TO THE BESSEL OPERATORS

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ABSTRACT. This paper provides a method to study the non-negativity of certain linear operators, from other operators with similar spectral properties. If these new operators are formally self-adjoint and non-negative, we can study the complex powers using an appropriate locally convex space. In this case, the initial operator also will be non-negative and we will be able to study their powers. In particular, we have applied this method to Bessel-type operators.

1. INTRODUCTION

Operators of type Bessel appear in the literature related with different versions of Hankel transform (see [1],[3],[6],[13]). We are going to consider Bessel operators on $(0, \infty)$ given by

$$(1) \quad \Delta_\mu = \frac{d^2}{dx^2} + (2\mu + 1)(x^{-1} \frac{d}{dx}) = x^{-2\mu-1} \frac{d}{dx} x^{2\mu+1} \frac{d}{dx}$$

and

$$(2) \quad S_\mu = \frac{d^2}{dx^2} - \frac{4\mu^2 - 1}{4x^2} = x^{-\mu-\frac{1}{2}} \frac{d}{dx} x^{2\mu+1} \frac{d}{dx} x^{-\mu-\frac{1}{2}},$$

which are related through

$$S_\mu = x^{\mu+\frac{1}{2}} \Delta_\mu x^{-\mu-\frac{1}{2}}.$$

This feature has inspired us to develop a method to study fractional powers based in a concept of "similar operator". Similar operators have the same spectral properties between them and of being non negative if one of them has this property. This method will apply in the contexts of Banach spaces and locally convex spaces.

Moreover, other feature of operators (1) and (2) is that one of these is selfadjoint and the other is not. To establish the complex powers of a differential operator in distributional spaces is important that this

1991 *Mathematics Subject Classification.* [2000]47A60, 26A33, 46F10; 46F12; 46F30.

Key words and phrases. Distributional operators, Hankel transform,.

operator be formally self-adjoint. It would therefore be interesting to obtain an operator similar and formally self-adjoint from a given initial operator.

To study the distributional fractional powers of Bessel operators, we use the theory developed by Martinez y Sanz in [7].

In section 2 we will review some of the standard facts about Hankel transform, convolution and Bessel operator in distributional and Lebesgue spaces, which are fundamental to establish the non negativity of Bessel operator. In Section 3 we will state our main results about similar operators in Banach spaces and we will describe the relation between their fractional powers. We extend this idea to locally convex spaces and we apply this ideas to the Bessel operator (1) and (2).

In section 4 we will establish the non-negativity of S_μ in suitable weighted Lebesgue spaces. In sections 5 and 6 we will establish the non negativity of S_μ in a suitable locally convex space and in its dual space.

For the convenience of the reader, we have added an Appendix with the proofs of some results about Hankel transform, convolutions, thus making our exposition self-contained.

2. PREVIOUS RESULTS ON HANKEL TRANSFORM AND CONVOLUTION

In this section we introduce the Lebesgue and distributional spaces necessary for our purposes.

By $\mathcal{D}(0, \infty)$ we denote the space of functions in $C^\infty(0, \infty)$ with compact support in $(0, \infty)$ with the usual topology, and by $\mathcal{D}'(0, \infty)$ the space of classical distributions in $(0, \infty)$.

Throughout this paper we assume $\mu > -\frac{1}{2}$. We will consider the Hankel transform defined in a suitable functional space denoted by \mathcal{H}_μ and given by

$$\mathcal{H}_\mu = \left\{ \phi \in C^\infty(0, \infty) : \sup_{x \in (0, \infty)} \left| x^m (x^{-1} D)^k x^{-\mu - \frac{1}{2}} \phi(x) \right| < \infty : m, k = 0, 1, 2, \dots \right\}$$

endowed with the family of seminorms $\{\gamma_{m,k}^\mu\}$, given by

$$(3) \quad \gamma_{m,k}^\mu(\phi) = \sup_{x \in (0, \infty)} \left| x^m (x^{-1} D)^k x^{-\mu - \frac{1}{2}} \phi(x) \right|,$$

\mathcal{H}_μ is a Fréchet space (see [13, Lemma 5.2-2, pp. 131]). Given $1 \leq p < \infty$ and a measurable function $w : (0, \infty) \rightarrow \mathbb{C}$ then we consider

$$(4) \quad L^p((0, \infty), w dx) = \left\{ f : (0, \infty) \rightarrow \mathbb{C} : f \text{ is measurable and } \int_0^\infty |f(x)|^p w(x) dx < \infty \right\}$$

where with dx we denote the Lebesgue usual measure. In $L^p((0, \infty), w dx)$ we consider the usual norm

$$\|f\|_{L^p((0, \infty), w dx)} = \left[\int_0^\infty |f(x)|^p w(x) dx \right]^{1/p}.$$

Moreover,

$$L^\infty((0, \infty), w) = \left\{ f : (0, \infty) \rightarrow \mathbb{C} : \text{measurable and } \text{ess sup}_{x \in (0, \infty)} |wf(x)| < \infty \right\}$$

endowed with the norm

$$\|f\|_{L^\infty((0, \infty), w)} = \|wf\|_\infty.$$

For simplicity of notation we write $L^p(w)$ and $L^\infty(w)$ instead of $L^p((0, \infty), w dx)$ and $L^\infty((0, \infty), w)$.

Let s and r as follow

$$(5) \quad r = x^{-\mu-\frac{1}{2}}$$

$$(6) \quad s = x^{2\mu+1}/c_\mu$$

with $c_\mu = 2^\mu \Gamma(\mu + 1)$.

Proposition 1. *It is verified that*

$$(7) \quad \mathcal{H}_\mu \subset L^1(sr) \cap L^\infty(r) \subset L^p(sr^p) \quad 1 \leq p < \infty,$$

with r and s given by (5) and (6).

Proof. The inclusion $\mathcal{H}_\mu \subset L^\infty(r)$ is immediate and also

$$(8) \quad \|\phi\|_{L^\infty(r)} = \gamma_{0,0}^\mu(\phi), \quad \phi \in \mathcal{H}_\mu.$$

It also verifies that $\mathcal{H}_\mu \subset L^1(sr)$ as

$$\int_0^\infty |\phi| sr dx = \int_0^1 |x^{-\mu-\frac{1}{2}} \phi| x^{2\mu+1} c_\mu^{-1} dx + \int_1^\infty x^m |x^{-\mu-\frac{1}{2}} \phi| x^{-m+2\mu+1} c_\mu^{-1} dx < \infty$$

if $m > 2\mu + 2$, and

$$(9) \quad \|\phi\|_{L^1(sr)} \leq C \{ \gamma_{0,0}^\mu(\phi) + \gamma_{m,0}^\mu(\phi) \}, \quad \phi \in \mathcal{H}_\mu$$

It also verifies that

$$(10) \quad \|\phi\|_{L^p(sr^p)} = \left\{ \int_0^\infty |\phi|^{p-1} r^{p-1} |\phi| r s \right\}^{\frac{1}{p}} \leq \left\{ \|\phi\|_{L^\infty(r)} \right\}^{\frac{p-1}{p}} \left\{ \|\phi\|_{L^1(sr)} \right\}^{\frac{1}{p}}$$

and by (8) and (9) we can consider a constant C' such that

$$(11) \quad \|\phi\|_{L^\infty(r)} \leq C' [\gamma_{0,0}^\mu(\phi) + \gamma_{m,0}^\mu(\phi)], \quad \phi \in \mathcal{H}_\mu.$$

$$(12) \quad \|\phi\|_{L^1(sr)} \leq C' [\gamma_{0,0}^\mu(\phi) + \gamma_{m,0}^\mu(\phi)], \quad \phi \in \mathcal{H}_\mu$$

and by (10), (11) and (12) we finally conclude that

$$(13) \quad \|\phi\|_{L^p(sr^p)} \leq C' [\gamma_{0,0}^\mu(\phi) + \gamma_{m,0}^\mu(\phi)], \quad \phi \in \mathcal{H}_\mu$$

□

If J_μ denote the Bessel function of first kind and order μ , we consider the Hankel transform h_μ given by

$$(14) \quad h_\mu \phi(x) = \int_0^\infty \sqrt{xy} J_\mu(xy) \phi(y) dy.$$

for $\phi \in \mathcal{H}_\mu$.

Remark 1. If $\phi \in L^1(sr)$ then Hankel transform $h_\mu \phi$ is well defined because the kernel $(xy)^{-\mu} J_\mu(xy)$ is bounded if $\mu > -\frac{1}{2}$ (see [11, (1), pp. 49]). By Proposition 1, $h_\mu \phi$ is well defined for all $\phi \in \mathcal{H}_\mu$ and is an automorphism (see [13, theorem 5.4-1, pp. 141]).

The space of the continuous linear functions $T : \mathcal{H}_\mu \rightarrow \mathbb{C}$ is denoted by \mathcal{H}'_μ .

Definition 1. We call a function $f \in L^1_{loc}((0, \infty))$ a *regular element* of \mathcal{H}'_μ if the identification

$$\begin{aligned} T_f : \mathcal{H}_\mu &\longrightarrow \mathbb{C} \\ \phi &\longrightarrow (T_f, \phi) = \int_0^\infty f \phi \end{aligned}$$

is well defined and is continuous; namely $T_f \in \mathcal{H}'_\mu$.

Remark 2. Given $T \in \mathcal{H}'_\mu$, we can consider the restriction of T to $\mathcal{D}(0, \infty)$ as a member of $\mathcal{D}'(0, \infty)$, because convergence in $\mathcal{D}(0, \infty)$ implies convergence in \mathcal{H}_μ . But $\mathcal{D}(0, \infty)$ is not dense in \mathcal{H}_μ (see [13]), consequently the behavior of an element $u \in \mathcal{H}'_\mu$ over $\mathcal{D}(0, \infty)$ not determines univocally the behavior of u as element of \mathcal{H}'_μ . If a locally integrable function is zero as regular element of \mathcal{H}'_μ , then is zero a.e in

$(0, \infty)$ because is zero as regular distribution in $\mathcal{D}'(0, \infty)$. So, regular distributions in \mathcal{H}'_μ are included in injective way in $\mathcal{D}'(0, \infty)$.

Proposition 2. Suppose that $1 \leq p < \infty$. A function in $L^p(sr^p)$ or $L^\infty(r)$ is a regular element of \mathcal{H}'_μ . In particular, the functions in \mathcal{H}_μ can be considered as regular elements of \mathcal{H}'_μ .

Proof. Let $f \in L^\infty(r)$, since $\mathcal{H}_\mu \subset L^1(sr) = L^1(r^{-1}/c_\mu)$, if $\phi \in \mathcal{H}_\mu$, then $\phi \in L^1(r^{-1})$ and $(T_f, \phi) = \int_0^\infty f\phi$ is well defined for $\phi \in \mathcal{H}_\mu$. So, by (9)

$$\begin{aligned} |(T_f, \phi)| &\leq \|f\|_{L^\infty(r)} \|\phi\|_{L^1(r^{-1})} = c_\mu \|f\|_{L^\infty(r)} \|\phi\|_{L^1(sr)} \\ &\leq Cc_\mu \|f\|_{L^\infty(r)} [\gamma_{0,0}^\mu(\phi) + \gamma_{m,0}^\mu(\phi)] \end{aligned}$$

consequently, f is a regular element of \mathcal{H}'_μ .

Now, let $f \in L^p(sr^p)$ with $1 \leq p < \infty$, then

(15)

$$|(T_f, \phi)| \leq \int_0^\infty |f\phi| = \int_0^\infty (r|f|)(s^{-1}r^{-1}|\phi|)s = \int_0^\infty (r|f|)(c_\mu r|\phi|)s$$

Since $r|f| \in L^p(s)$ and $r|\phi| \in L^q(s)$ because $\phi \in \mathcal{H}_\mu \subset L^q(sr^q)$ (see (7)), being q the conjugate of p , we obtain by Hölder inequality and (13)

$$|(T_f, \phi)| \leq c_\mu \|f\|_{L^p(sr^p)} \|\phi\|_{L^q(sr^q)} \leq Cc_\mu \|f\|_{L^p(sr^p)} [\gamma_{0,0}^\mu(\phi) + \gamma_{m,0}^\mu(\phi)]$$

with m a positive integer $m > 2\mu + 2$. So, f is a regular element of \mathcal{H}'_μ . \square

Given f, g defined in $(0, \infty)$, the Hankel convolution $f \sharp g$ is defined formally by

$$(16) \quad (f \sharp g)(x) = \int_0^\infty \int_0^\infty D_\mu(x, y, z) f(y) g(z) dy dz$$

where $D_\mu(x, y, z)$ is given by

(17)

$$D_\mu(x, y, z) = \begin{cases} \frac{2^{\mu-1}(xyz)^{-\mu+\frac{1}{2}}}{\Gamma(\mu+\frac{1}{2})\sqrt{\pi}} (A(x, y, z))^{2\mu-1} & \text{si } |x-y| < z < x+y \\ 0 & \text{si } 0 < z < |x-y| \text{ o } z > x+y. \end{cases}$$

$A(x, y, z)$ is the measure of area of the triangle with sides x, y, z and $|x-y| < z < x+y$ is the condition for such a triangle to exist, and in this case $A(x, y, z) = \frac{1}{4}\sqrt{[(x+y)^2 - z^2][z^2 - (x-y)^2]}$.

The proofs of Theorems 1 and 2 and Proposition 3 are displayed in Appendix.

Theorem 1. *Let $f \in L^1(sr)$.*

- (1) *If $g \in L^\infty(r)$, then the convolution $f \sharp g(x)$ exists for every $x \in (0, \infty)$, $f \sharp g \in L^\infty(r)$ and*

$$(18) \quad \|f \sharp g\|_{L^\infty(r)} \leq \|f\|_{L^1(sr)} \|g\|_{L^\infty(r)}.$$

- (2) *If $g \in L^p(sr^p)$ ($1 \leq p < \infty$), then the convolution $f \sharp g(x)$ exists for a. e. $x \in (0, \infty)$, $f \sharp g \in L^p(sr^p)$ and*

$$(19) \quad \|f \sharp g\|_{L^p(sr^p)} \leq \|f\|_{L^1(sr)} \|g\|_{L^p(sr^p)}.$$

Theorem 2. *Let $\{\phi_n\} \subset L^1(rs)$ such that*

- (1) $\phi_n \geq 0$ in $(0, \infty)$,
- (2) $\int_0^\infty \phi_n(x)r(x)s(x) dx = 1$ for all n
- (3) For $\delta > 0$, $\lim_{n \rightarrow \infty} \int_\delta^\infty \phi_n(x)r(x)s(x) dx = 0$.

Let $f \in L^\infty(r)$ and continuous in $x_0 \in (0, \infty)$, then $\lim_{n \rightarrow \infty} f \sharp \phi_n(x_0) = f(x_0)$. Further, if rf is uniformly continuous in $(0, \infty)$ then $\lim_{n \rightarrow \infty} \|f \sharp \phi_n(x) - f(x)\|_{L^\infty(r)} = 0$

Proposition 3. *Let $f, g \in L^1(sr)$, then*

$$(20) \quad h_\mu(f \sharp g) = rh_\mu(f)h_\mu(g).$$

2.1. The Bessel operator S_μ . In this section we summarize some elementary properties of S_μ on the spaces \mathcal{H}_μ and \mathcal{H}'_μ . For most of the proofs we refer the reader to [13].

Proposition 4. (1) *The operator $S_\mu : \mathcal{H}_\mu \rightarrow \mathcal{H}_\mu$ is continuous.*

- (2) *If $\lambda \geq 0$, the operator*

$$\begin{array}{ccc} \mathcal{H}_\mu & \longrightarrow & \mathcal{H}_\mu \\ \phi & \longrightarrow & (\lambda + x^2)\phi \end{array}$$

is continuous.

- (3) *If $\lambda > 0$, the operator*

$$\begin{array}{ccc} \mathcal{H}_\mu & \longrightarrow & \mathcal{H}_\mu \\ \phi & \longrightarrow & (\lambda + x^2)^{-1}\phi \end{array}$$

is continuous.

Proposition 5. *Let $\phi \in \mathcal{H}_\mu$, then*

- (1) $(h_\mu S_\mu \phi)(x) = -x^2 (h_\mu \phi)(x), \quad x \in (0, \infty).$
- (2) $(S_\mu h_\mu \phi) = h_\mu(-y^2 \phi(y)).$

Proposition 6. *The following continuous operators in \mathcal{H}_μ can be extended to \mathcal{H}'_μ in the following way:*

- (1) *The Hankel transform h_μ*

$$(h_\mu u, \phi) = (u, h_\mu \phi), \quad u \in \mathcal{H}'_\mu, \phi \in \mathcal{H}_\mu,$$

and $h_\mu : \mathcal{H}'_\mu \rightarrow \mathcal{H}'_\mu$ is a bijective mapping .

- (2) *The differential operator S_μ*

$$(S_\mu u, \phi) = (u, S_\mu \phi), \quad u \in \mathcal{H}'_\mu, \phi \in \mathcal{H}_\mu.$$

- (3) *The product of $(\lambda + x^2)$ for $\lambda \geq 0$*

$$((\lambda + x^2)u, \phi) = (u, (\lambda + x^2)\phi), \quad u \in \mathcal{H}'_\mu, \phi \in \mathcal{H}_\mu.$$

- (4) *The product of $(\lambda + x^2)^{-1}$ for $\lambda > 0$*

$$((\lambda + x^2)^{-1}u, \phi) = (u, (\lambda + x^2)^{-1}\phi), \quad u \in \mathcal{H}'_\mu, \phi \in \mathcal{H}_\mu.$$

Proposition 7. *If $u \in \mathcal{H}'_\mu$, then*

- (1) $h_\mu S_\mu u = -x^2 h_\mu u$
- (2) $S_\mu h_\mu u = h_\mu(-y^2 u)$

Proposition 8. *The following equalities are valid in \mathcal{H}_μ and \mathcal{H}'_μ for $n = 1, 2, \dots$*

- (1) $(-S_\mu + \lambda)^n h_\mu = h_\mu(y^2 + \lambda)^n.$

Moreover if $\lambda > 0$

- (2) $h_\mu(-S_\mu + \lambda)^{-n} = (y^2 + \lambda)^{-n} h_\mu.$
- (3) $h_\mu(-S_\mu(-S_\mu + \lambda)^{-1})^n = y^{2n}(y^2 + \lambda)^{-n} h_\mu$

Proof. (1) is immediate consequence of item (2) of Proposition 7 .

Noting that $h_\mu(y^2 + \lambda)^{-1} h_\mu$ is inverse operator of $-S_\mu + \lambda$, then (2) is obtained with a simple application of Proposition 14 (see Appendix) and induction over n .

Equality (3) follows immediately by item (2) of Proposition 7 and induction over n . \square

3. PREVIOUS RESULTS ABOUT SIMILAR OPERATORS AND NO-NEGATIVITY

Let X and Y be Banach spaces. Suppose that there is an isometric isomorphism $T : X \rightarrow Y$ and let $T^{-1} : Y \rightarrow X$ its inverse. Let A a

linear operator $A : D(A) \subset X \rightarrow X$ then we can consider the operator $B = TAT^{-1}$, $B : D(B) \subset Y \rightarrow Y$ with domain

$$D(B) = \{x \in Y : T^{-1}x \in D(A)\},$$

Under these conditions we will say that such operators are *similar*.

Proposition 9. *Let A and B similar operators. Then A is non negative if and only if so is B .*

Proof. Let $B = TAT^{-1}$, the proof is immediate noting that

$$(zId - B)^{-1} = T(zId - A)^{-1}T^{-1},$$

for a complex number z . □

If A is a non negative operator, then for $\alpha \in \mathbb{C}$ such that $\operatorname{Re} \alpha > 0$, $n > \operatorname{Re} \alpha$ and $\phi \in D(A^n)$, the Balakrishnan operator associated with A , can be represented by

$$J_A^\alpha \phi = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty \lambda^{\alpha-1} [A(\lambda + A)^{-1}]^m \phi d\lambda,$$

(see [7, Proposition 3.1.3, pp.59]).

If A is bounded, J_A^α can be consider as fractional power of A , and in other case we can consider the following representation for fractional power given in [7, Theorem 5.2.1, pp.114]),

$$A^\alpha = (A + \lambda)^n J_A^\alpha (A + \lambda)^{-n},$$

with α, n as above and $\lambda \in \rho(-A)$.

When two operators are similar, the fractional powers also meet this property. Thus, we have the following result:

Proposition 10. *Let A and B similar and non negative operators. If $\alpha \in \mathbb{C}$ such that $\operatorname{Re} \alpha > 0$, then*

$$(21) \quad J_B^\alpha = T J_A^\alpha T^{-1},$$

and

$$(22) \quad B^\alpha = T A^\alpha T^{-1},$$

where T is the isometric isomorphism that verifies $B = TAT^{-1}$.

Proof. We observe that if $B = TAT^{-1}$ then $B^n = T A^n T^{-1}$ and

$$D(T J_A^\alpha T^{-1}) = \{x \in Y : T^{-1}x \in D(J_A^\alpha)\} = \{x \in Y : T^{-1}x \in D(A^n)\} = D(J_B^\alpha),$$

and (21) is immediate from properties of Bochner integral. In (22) the equality of domains is evident and

$$B^\alpha = (B + \lambda)^n J_B^\alpha (B + \lambda)^{-n} = (TAT^{-1} + \lambda)^n T J_A^\alpha T^{-1} (TAT^{-1} + \lambda)^{-n} =$$

$$= T(A + \lambda)^n T^{-1} T J_A^\alpha T^{-1} T(A + \lambda)^{-n} T^{-1} = T A^\alpha T^{-1}.$$

□

Now, we consider a Hausdorff locally convex space X with a direct family of seminorms $\{\|\cdot\|_{X,\nu}\}_{\nu \in \mathcal{A}}$. Let Y be a linear space such that there is a linear isomorphism $L : X \rightarrow Y$. Then, we can define the following family of seminorms in Y :

$$\|y\|_{Y,\nu} = \|L^{-1}(y)\|_{X,\nu}.$$

Thus, the linear space Y with the directed family of seminorms $\{\|\cdot\|_{Y,\nu}\}_{\nu \in \mathcal{A}}$ is a Hausdorff locally convex space and X and Y are isomorphic. The propositions 9 and 10 can be easily extended to the case of non negative operators in locally convex spaces.

3.1. Applications to Bessel operator. Given $\mu > -\frac{1}{2}$, we consider the differential operator given by (1) defined in $(0, \infty)$.

We are now going to apply the observations considered in the previous section to the operator Δ_μ . First, we calculate the Sturm-Liouville form of Δ_μ , thereby obtaining the operator

$$T_\mu = x^{2\mu+1} \Delta_\mu$$

which is formally self-adjoint. Operators of type $fT_\mu f$, with $f \in C^\infty(0, \infty)$, are still formally self-adjoint. If we want the new operator to be similar to Δ_μ , namely type $r^{-1} \Delta_\mu r$, we have to consider $r = x^{-\mu-\frac{1}{2}}$. Thus the operator

$$(23) \quad S_\mu = x^{\mu+\frac{1}{2}} \Delta_\mu x^{-\mu-\frac{1}{2}}$$

is formally self-adjoint and similar to Δ_μ and hence with the same spectral properties.

Since mappings $L_r : L^p(r^p s) \rightarrow L^p(s)$ with $1 \leq p < \infty$ (or $L_r : L^\infty(r) \rightarrow L^\infty$) given by $L_r(f) = rf$ are isometric isomorphisms, if we consider the part of the distributional operator Δ_μ in the spaces $L^p(s)$ (or L^∞), i.e., the operator with domain

$$D((\Delta_\mu)_{L^p(s)}) = \{f \in L^p(s) : \Delta_\mu f \in L^p(s)\},$$

and given by $(\Delta_\mu)_{L^p(s)} f = \Delta_\mu f$. Then, applying the ideas developed in the previous section, it is enough to study the operator S_μ in the spaces $L^p(sr^p)$ (or $L^\infty(r)$).

4. FRACTIONAL POWERS OF S_μ IN LEBESGUE SPACES

Now, we study the non negativity of the part in $L^p(sr^p)$ and in $L^\infty(r)$ of distributional differential operator S_μ given by (2).

Let $1 \leq p < \infty$. We will denote by $S_{\mu,p}$ the part of S_μ in $L^p(sr^p)$; i.e. the operator S_μ with domain

$$D(S_{\mu,p}) = \{f \in L^p(sr^p) : S_\mu f \in L^p(sr^p)\}$$

and given by $S_{\mu,p}f = S_\mu f$.

Analogously, by $S_{\mu,\infty}$ we will denote the part of S_μ in $L^\infty(r)$; namely, the operator S_μ with domain

$$D(S_{\mu,\infty}) = \{f \in L^\infty(r) : S_\mu f \in L^\infty(r)\}$$

and $S_{\mu,\infty}f = S_\mu f$.

In order to study the non negativity of operators $-S_{\mu,\infty}$ and $-S_{\mu,p}$ we consider the following function:

$$(24) \quad \mathcal{K}_\nu(x) = \frac{1}{2} \left(\frac{1}{2}x \right)^\nu \int_0^\infty e^{-t - \frac{x^2}{4t}} \frac{dt}{t^{\nu+1}},$$

for $x \in (0, \infty)$. Since for $\nu < 0$

$$\int_0^\infty e^{-t - \frac{x^2}{4t}} t^{-\nu-1} dt < \int_0^\infty e^{-t} t^{-\nu-1} dt < \infty$$

and for $\nu \geq 0$ the function $e^{-t - \frac{x^2}{4t}} t^{-\nu-1}$ is bounded in a neighborhood of zero, \mathcal{K}_ν is well defined for $\nu \in \mathbb{R}$ and $\mathcal{K}_\nu > 0$.

Remark 3. For non-integer values of ν , \mathcal{K}_ν (see [11, (15), pp. 183]), coincides with the Macdonald's function K_ν (see [11, (6) and (7), pp. 78]) given by

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi} \quad x > 0,$$

with I_ν is the modified Bessel function over $(0, \infty)$ (see [11, (2), pp. 77]). For integers values of ν , K_ν is defined by

$$K_n(x) = \lim_{\nu \rightarrow n} K_\nu(x) \quad x > 0.$$

Now, given $\lambda > 0$, we consider the function

$$N_\lambda(x) = \lambda^{\frac{\mu}{2}} x^{\frac{1}{2}} \mathcal{K}_\mu(\sqrt{\lambda} x), \quad x \in (0, \infty).$$

The following lemmas describe properties of the kernel N_λ which are crucial for the study of the no-negativity of Bessel operator (for proofs see Appendix).

Lemma 1. *Given $\mu > -\frac{1}{2}$ and $\lambda > 0$ then*

$$a) N_\lambda \in L^1(sr) = L^1\left(\frac{r^{-1}}{c_\mu}\right) \text{ and}$$

$$\|N_\lambda\|_{L^1(sr)} = \frac{1}{\lambda}.$$

b)

$$h_\mu N_\lambda(y) = \frac{y^{\mu+\frac{1}{2}}}{\lambda + y^2}.$$

Lemma 2. *Let $1 \leq p < \infty$. If $f \in L^p(sr^p)$ or $L^\infty(r)$ then the following equality holds on \mathcal{H}'_μ*

$$(25) \quad h_\mu(N_\lambda \sharp f) = \frac{1}{\lambda + y^2} h_\mu(f)$$

Finally, we can establish the main result of this section.

Theorem 3. *Given $\mu > -\frac{1}{2}$, then*

- (1) *The operators $S_{\mu,p}$ and $S_{\mu,\infty}$ are closed .*
- (2) *The operators $-S_{\mu,p}$ and $-S_{\mu,\infty}$ are non-negative.*

Proof. (1) Let $\{f_n\}_{n=1}^\infty \subset D(S_{\mu,\infty})$ such that

$$\lim_{n \rightarrow \infty} f_n = f \quad \text{and} \quad \lim_{n \rightarrow \infty} S_{\mu,\infty} f_n = g$$

in $L^\infty(r)$. Since convergence in $L^\infty(r)$ implies convergence in $\mathcal{D}'(0, \infty)$, then given $\phi \in \mathcal{D}(0, \infty)$

$$(S_\mu f, \phi) = (f, S_\mu \phi) = \lim_{n \rightarrow \infty} (f_n, S_\mu \phi) = \lim_{n \rightarrow \infty} (S_\mu f_n, \phi) = (g, \phi),$$

so, $S_\mu f = g$ and $S_{\mu,\infty}$ is closed. The case of $S_{\mu,p}$ is similar.

(2) Let $\lambda > 0$ and $f \in D(S_{\mu,\infty})$ such that $(\lambda - S_{\mu,\infty}) f = 0$. Then $(\lambda - S_{\mu,\infty}) f \in L^\infty(r)$ and is null as regular element of \mathcal{H}'_μ , so

$$h_\mu(\lambda - S_{\mu,\infty} f) = 0$$

in \mathcal{H}'_μ . By Proposition 7, we obtain that

$$(\lambda + y^2) h_\mu f = 0$$

in \mathcal{H}'_μ , and hence by Proposition 6

$$h_\mu f = (\lambda + y^2)^{-1}(\lambda + y^2)h_\mu f = 0.$$

Then, $f = 0$ as element of \mathcal{H}'_μ and by Remark 2 we conclude that $f = 0$ a.e in $(0, \infty)$ and, $\lambda - S_{\mu, \infty}$ is injective. Now, let $f \in L^\infty(r)$ and $g = N_\lambda \# f$. Then, by Theorem 1, $g \in L^\infty(r)$ and

$$h_\mu((\lambda - S_{\mu, \infty})g) = (\lambda + y^2)h_\mu g = (\lambda + y^2)h_\mu(N_\lambda \# f) = h_\mu f.$$

By injectivity of Hankel transform in \mathcal{H}'_μ we obtain that

$$(\lambda - S_{\mu, \infty})g = f,$$

so, $\lambda - S_{\mu, \infty}$ is onto. Also,

$$\|(\lambda - S_{\mu, \infty})^{-1}f\|_{L^\infty(r)} = \|g\|_{L^\infty(r)} = \|N_\lambda \# f\|_{L^\infty(r)} \leq \|N_\lambda\|_{L^1(rs)} \|f\|_{L^\infty(r)} = \frac{1}{\lambda} \|f\|_{L^\infty(r)}$$

hence $-S_{\mu, \infty}$ is non-negative. The proof of non-negativity of $-S_{\mu, p}$ is similar. \square

Remark 4. : In [9], the result of theorem above has been obtained for the particular case $p = 2$ and in \mathbb{R}_+^n .

Now, in view of non-negativity of $-S_{\mu, \infty}$ and $-S_{\mu, p}$ we can consider the complex fractional powers. If $\alpha \in \mathbb{C}$, $\text{Re } \alpha > 0$ and $n > \text{Re } \alpha$ then the fractional power of $-S_{\mu, \infty}$ can be represented by:

$$(-S_{\mu, \infty})^\alpha = (-S_{\mu, \infty} + 1)^n \mathcal{J}_\infty^\alpha (-S_{\mu, \infty} + 1)^{-n},$$

(see [7, (5.20), pp. 114]), where with $\mathcal{J}_\infty^\alpha$ we denote the Balakrishnan operator associated to $-S_{\mu, \infty}$ given by:

$$\mathcal{J}_\infty^\alpha \phi = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^\infty \lambda^{\alpha-1} \left[-S_{\mu, \infty}(\lambda - S_{\mu, \infty})^{-1} \right]^n \phi \, d\lambda,$$

for α and n in the previous conditions and $\phi \in D((-S_{\mu, \infty})^n)$, (see [7, (3.4), pp. 59]). The case of $(-S_{\mu, p})^\alpha$ is analogous.

5. NONNEGATIVITY OF BESSEL OPERATOR S_μ IN THE SPACE \mathcal{B}

In order to study non-negativity of Bessel operator in a locally convex space, we begin with the following observation:

Remark 5. *The continuous operator $-S_\mu : \mathcal{H}_\mu \rightarrow \mathcal{H}_\mu$ is not non-negative.*

Indeed, if we suppose that $-S_\mu$ is non-negative in \mathcal{H}_μ , by the continuity of $-S_\mu$ in \mathcal{H}_μ , given $\alpha \in \mathbb{C}$, $0 < \alpha < 1$ and according to [7,

Chapter 5, pp. 105 and 134]), we have that fractional power $(-S_\mu)^\alpha$ would be given by

$$(26) \quad (-S_\mu)^\alpha \phi = \frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{\alpha-1} (-S_\mu)(\lambda - S_\mu)^{-1} \phi \, d\lambda$$

and $D((-S_\mu)^\alpha) = D(-S_\mu) = \mathcal{H}_\mu$. Applying the Hankel transform in (26) we obtain

$$\begin{aligned} h_\mu \left((-S_\mu)^\alpha \phi \right) (y) &= \frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{\alpha-1} y^2 (\lambda + y^2)^{-1} h_\mu \phi(y) \, d\lambda. \\ &= (y^2)^\alpha h_\mu \phi(y), \end{aligned}$$

where we have used equality (3) of Proposition 8 and [7, Remark 3.1.1]). In this case it would mean that $(y^2)^\alpha h_\mu \phi(y) \in \mathcal{H}_\mu$ which is false in general (just consider $\phi(y) = y^{\mu+\frac{1}{2}} e^{-y^2}$ and $\alpha = \frac{1}{4}$).

Now, we consider the Banach space $Y = L^1(sr) \cap L^\infty(r)$ with the norm

$$\|f\|_Y = \max \left(\|f\|_{L^1(sr)}, \|f\|_{L^\infty(r)} \right),$$

and the part of the Bessel operator in Y , $(S_\mu)_Y$, with domain

$$D[(S_\mu)_Y] = \{f \in Y : S_\mu f \in Y\}.$$

From Theorem 3 it is evident that $-(S_\mu)_Y$ is closed and nonnegative. We have the following proposition:

Proposition 11. $D[(S_\mu)_Y] \subset C_0(0, \infty)$.

Proof. By (7) and (38), $L^1(sr) \cap L^\infty(r) \subset L^1(0, \infty) \cap L^2(0, \infty)$, then for $f \in D[(S_\mu)_Y]$, f and $S_\mu f$ are in $L^1(0, \infty)$. By Remark 11 (see Appendix) then $h_\mu f - h_\mu S_\mu f$ are in $L^\infty(0, \infty)$. By (1) of Proposition 7 we have that

$$(1 + y^2) |h_\mu f| \leq M,$$

so, $h_\mu f \in L^1(0, \infty)$.

We have thus proved that for $f \in D[(S_\mu)_Y]$ then f and $h_\mu(f)$ are in $L^1(0, \infty) \cap L^2(0, \infty)$. Then, by Remark 10 (see Appendix), we obtain that

$$h_\mu(h_\mu(f))(x) = f(x), \quad \text{a.e. } x \in (0, \infty).$$

Since $h_\mu(f) \in L^1(0, \infty)$ then by Proposition 15, $f = g$ a.e in $(0, \infty)$ with $g \in C_0(0, \infty)$. \square

Now, we consider the following space:

$$\mathcal{B} = \{f \in Y : (S_\mu)^k f \in Y \quad \text{for } k = 0, 1, 2, \dots\} = \bigcap_{k=0}^{\infty} D[((S_\mu)_Y)^k],$$

with the seminorms

$$\rho_m(f) = \max_{0 \leq k \leq m} (\|(S_\mu)^k f\|_Y), \quad m = 0, 1, 2, \dots$$

Remark 6. From Proposition 11 it is evident that $\mathcal{B} \subset C^\infty(0, \infty) \cap C_0(0, \infty)$. Moreover, is clear from (7) that $\mathcal{B} \subset L^p(sr^p)$ for all $1 \leq p < \infty$, and considering (1) of proposition 4 we have that $\mathcal{H}_\mu \subset \mathcal{B}$ and the topology of \mathcal{H}_μ induced by \mathcal{B} is weaker than the usual topology given in section 2. Indeed, from (11) and (12) we have that

$$(27) \quad \|\phi\|_Y \leq C [\gamma_{0,0}^\mu(\phi) + \gamma_{k,0}^\mu(\phi)], \quad \phi \in \mathcal{H}_\mu$$

for $k > 2\mu + 2$, and by continuity of S_μ in \mathcal{H}_μ , we deduce that given a seminorm ρ_m there exists a finite set of seminorms $\{\gamma_{m_i, k_i}^\mu\}_{i=1}^r$ and constants c_1, \dots, c_r such that

$$\rho_m(\phi) \leq \sum c_i \gamma_{m_i, k_i}^\mu(\phi), \quad \phi \in \mathcal{H}_\mu.$$

Moreover, \mathcal{H}_μ is dense in \mathcal{B} because $\mathcal{D}(0, \infty)$ it is.

Proposition 12. \mathcal{B} is not normable.

Proof. Suppose that \mathcal{B} is locally bounded, then there exists an integer positive n such that the set

$$V_n = \left\{ \phi \in \mathcal{B} : \rho_n(\phi) < \frac{1}{n} \right\},$$

are bounded. Then, there exists a $t_n > 0$ such that

$$(28) \quad V_n \subset t_n V_{n+1}.$$

Let $\phi \in \mathcal{B}$ and $\varphi = ((n+1)\rho_n(\phi))^{-1}\phi$. Then $\varphi \in V_n$ and by (28) $(t_n)^{-1}\varphi \in V_{n+1}$, and hence $\rho_{n+1}((t_n)^{-1}\varphi) < \frac{1}{n+1}$, so

$$(29) \quad \rho_{n+1}(\phi) \leq t_n \rho_n(\phi)$$

Given a constant $l > 0$ and $f, g \in C^{2k}(0, \infty)$ related by $f(x) = g(lx)$, we have that

$$(30) \quad (S_\mu)^k f(x) = (l^2)^k ((S_\mu)^k g)(lx).$$

Now, let $\phi \in \mathcal{B}$ such that $(S_\mu)^{n+1}\phi$ non-identically vanishing function and a constant $s > 1$. Then $\psi(x) = \phi(s^{-1}x)$ remains in \mathcal{B} and verified

that $(S_\mu)^{n+1}\psi$ is a non-identically vanishing function and $\phi(x) = \psi(sx)$. Then,

$$(31) \quad \begin{aligned} \|(S_\mu)^{n+1}\psi\|_{L^\infty(r)} &= s^{-\mu-\frac{1}{2}}s^{-2(n+1)}\|(S_\mu)^{n+1}\phi\|_{L^\infty(r)} \leq \\ s^{-\mu-\frac{1}{2}}s^{-2(n+1)}t_n\rho_n(\phi) &\leq s^{-\mu-\frac{1}{2}}s^{-2(n+1)}s^{\mu+\frac{1}{2}}s^{2n}t_n\rho_n(\psi) = s^{-2}t_n\rho_n(\psi). \end{aligned}$$

Since (31) is verified for all $s > 1$, taking $s \rightarrow \infty$, we conclude that $\|(S_\mu)^{n+1}\psi\|_{L^\infty(r)} = 0$ which contradicts the assumption about ψ . Then the proposition follows. \square

We denote with $(S_\mu)_\mathcal{B}$ the part of Bessel operator S_μ in \mathcal{B} . By definition of \mathcal{B} , it is evident that the domain of $(S_\mu)_\mathcal{B}$ is \mathcal{B} and is verified the following result

Theorem 4. *\mathcal{B} is a Fréchet space and $-(S_\mu)_\mathcal{B}$ is continuous and non-negative operator on \mathcal{B} .*

Proof. The proof is immediate by Proposition 1.4.2 given in [7]. \square

6. NONNEGATIVITY OF BESSEL OPERATOR S_μ IN THE DISTRIBUTIONAL SPACE \mathcal{B}'

In this section we study the non-negativity of Bessel operator in the topological dual space of \mathcal{B} with the strong topology, i.e the space \mathcal{B}' with the seminorms $\{|\cdot|_B\}$, where the sets B are in the family of bounded sets in \mathcal{B} , and are given by

$$|T|_B = \sup_{\phi \in B} |(T, \phi)|, \quad T \in \mathcal{B}'.$$

Remark 7. *As in [8, Remark 3.4, pp.263], \mathcal{B}' is sequentially complete because \mathcal{B} is non-normable. Moreover, and for $1 \leq p \leq \infty$ then $L^p(sr^p) \subset \mathcal{B}'$. To prove this, we observe that given $f \in L^p(sr^p)$ and $\phi \in \mathcal{B}$ and q the conjugate of p then*

$$(32) \quad \left| \int_0^\infty f\phi \right| = \left| \int_0^\infty f\phi s^{-1}r^{-p}sr^p \right| \leq \|f\|_{L^p(sr^p)} \|\phi s^{-1}r^{-p}\|_{L^q(sr^p)},$$

and

$$(33) \quad \begin{aligned} \|\phi s^{-1}r^{-p}\|_{L^q(sr^p)} &= \left\{ \int_0^\infty |\phi s^{-1}r^{-p}|^q sr^p \right\}^{\frac{1}{q}} = \left\{ \int_0^\infty |\phi|^q (c_\mu r^2 r^{-p})^q sr^p \right\}^{\frac{1}{q}} = \\ &= c_\mu \left\{ \int_0^\infty |\phi|^q r^{2q-pq+p} s \right\}^{\frac{1}{q}} = c_\mu \left\{ \int_0^\infty |\phi|^q sr^q \right\}^{\frac{1}{q}}. \end{aligned}$$

Moreover, by (10) we have that

$$(34) \quad \|\phi\|_{L^q(sr^q)} \leq \rho_0(\phi),$$

and from (32), (33) and (34) we obtain that $f \in \mathcal{B}'$.

Now, let B be a bounded set in \mathcal{B} then

$$\sup_{\phi \in B} \left| \int_0^\infty f \phi \right| \leq c_\mu \|f\|_{L^p(sr^p)} \sup_{\phi \in B} \|\phi\|_{L^q(sr^q)} \leq c_\mu \|f\|_{L^p(sr^p)} \sup_{\phi \in B} \rho_0(\phi).$$

Consequently, the topology in $L^p(sr^p)$ induced by \mathcal{B}' with strong topology is weaker than the usual topology.

Remark 8. By Remark 6, $\mathcal{B}' \subset \mathcal{H}'_\mu$. Moreover, from the continuity of the Bessel operator in \mathcal{B} , we can consider S_μ in \mathcal{B}' as adjoint operator of S_μ in \mathcal{B} , that is

$$(S_\mu T, \phi) = (T, S_\mu \phi), \quad T \in \mathcal{B}', \phi \in \mathcal{B},$$

and we denote with $(S_\mu)_{\mathcal{B}'}$ the part of Bessel operator in \mathcal{B}'

Theorem 5. The operator $-(S_\mu)_{\mathcal{B}'}$ is continuous and non negative considering the strong topology in \mathcal{B}' .

Proof. The proof of continuity is identical to the proof given in [8, Theorem 3.5, pp. 264] for the Laplacean operator and the non negativity is a consequence of theory of fractional powers in distributional spaces (see [7, pp. 24]). \square

Remark 9. The operator $(S_\mu)_{\mathcal{B}'}$ is not injective because the function $x^{\mu+\frac{1}{2}}$ is solution of $S_\mu = 0$ and belongs to \mathcal{B}' , in fact

$$|(x^{\mu+\frac{1}{2}}, \phi)| \leq c_\mu \|\phi\|_{L^1(sr)} \leq c_\mu \rho_0(\phi), \quad (\phi \in \mathcal{B}).$$

According to representation of fractional powers of operators in locally convex spaces given in [7], for $\operatorname{Re} \alpha > 0$, $n > \operatorname{Re} \alpha$, $T \in \mathcal{B}'$, $(-(S_\mu)_{\mathcal{B}'})^\alpha$ is given by

$$(-(S_\mu)_{\mathcal{B}'})^\alpha T = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^\infty \lambda^{\alpha-1} \left[-(S_\mu)_{\mathcal{B}'} (\lambda - (S_\mu)_{\mathcal{B}'})^{-1} \right]^n T d\lambda.$$

From the general theory of fractional power in sequentially complete locally convex spaces (see [7, pp. 134]), we deduce immediately the properties of powers as multiplicativity, spectral mapping theorem, and

- 1) If $\operatorname{Re} \alpha > 0$ then

$$(35) \quad \left((-S_\mu)_B^\alpha \right)^* = \left((-S_\mu)_B^* \right)^\alpha.$$

Since $(-S_\mu)_B^* = -(S_\mu)_{B'}$ then from (35) we obtain the following duality formula

$$((-(S_\mu)_{B'})^\alpha T, \phi) = (T, (-(S_\mu)_B)^\alpha \phi), \quad (\phi \in \mathcal{B}, T \in \mathcal{B}').$$

2) Since the usual topology in $L^p(sr^p)$ is stronger than the topology induced by \mathcal{B}' then we can deduce that

$$[(-(S_\mu)_{B'})^\alpha]_{L^p(sr^p)} = ((-(S_\mu)_B)^\alpha),$$

if $\operatorname{Re} \alpha > 0$, (see [7, Theorem 12.1.6, pp. 284]).

7. APPENDIX

7.1. Some properties of Hankel transform in Lebesgue spaces.

Proposition 13. *If $f, g \in L^1(sr)$, then:*

- (1) $h_\mu f \in L^\infty(r)$.
- (2)

$$(36) \quad \int_0^\infty h_\mu f g = \int_0^\infty f h_\mu g.$$

Proof. The assertion (1) follows from the equalities

$$x^{-\mu-\frac{1}{2}} h_\mu(f)(x) = x^{-\mu-\frac{1}{2}} \int_0^\infty \sqrt{xy} J_\mu(xy) f(y) dy = \int_0^\infty (xy)^{-\mu} J_\mu(xy) f(y) y^{\mu+\frac{1}{2}} dy.$$

The existence and equality of the integrals in (36) are an immediate consequence of Tonelli-Hobson theorem. \square

In [3] is studied a version of Hankel transform given by:

$$H_\mu(f)(x) = c_\mu \int_0^\infty (xy)^{-\mu} J_\mu(xy) f(y) s(y) dy,$$

for $f \in L^1(s)$. H_μ is related whit h_μ by

$$h_\mu(f) = r^{-1} H_\mu(rf).$$

for $f \in L^1(sr)$ (r and s like as in section 2). From this relation and the inversion theorem for H_μ (see [3, Corollary 2e, pp 316]) , we obtain the following inversion theorem for h_μ

Proposition 14. *If $f \in L^1(sr)$ and $h_\mu(f) \in L^1(sr)$ then f may be redefined on a set of measure zero so that it is continuous in $(0, \infty)$ and*

$$(37) \quad f(x) = \int_0^\infty \sqrt{xy} J_\mu(xy) h_\mu(f)(y) dy = h_\mu(h_\mu(f))(x)$$

Remark 10. *From the above Proposition we deduce immediately the validity of equality $h_\mu h_\mu f = f$ in \mathcal{H}_μ and \mathcal{H}'_μ .*

With $L^p(0, \infty)$ we denote the usual Lebesgue space given by (4) with $w(x) = 1$.

Remark 11. *Since the function $(z)^{\frac{1}{2}} J_\mu(z)$ is bounded in $(0, \infty)$ for $\mu > -\frac{1}{2}$, then for $f \in L^1(0, \infty)$ we have that $h_\mu f$ is continuous and $\|h_\mu f\|_\infty \leq C\|f\|_1$.*

As usual, we denote with $C_0(0, \infty)$ the set of continuous functions in $(0, \infty)$ and vanishes at infinity. We have the following proposition:

Proposition 15. $h_\mu(L^1(0, \infty)) \subset C_0(0, \infty)$

Proof. First, we observe that

$$(38) \quad L^1(sr) \cap L^\infty(r) \subset L^1(0, \infty).$$

Indeed,

$$\begin{aligned} \int_0^\infty |f| dx &= \int_0^\infty |f| r r^{-1} dx = \int_0^1 |f| r r^{-1} dx + \int_1^\infty |f| r r^{-1} dx \leq \\ &\leq \|f\|_{L^\infty(r)} \int_0^1 r^{-1} dx + \int_1^\infty |f| r^{-1} dx = C\|f\|_{L^\infty(r)} + c_\mu \|f\|_{L^1(rs)}, \end{aligned}$$

because $r < 1$ in $[1, \infty)$, $\mu + \frac{1}{2} > 0$ and $rs = c_\mu^{-1} r^{-1}$.

By (38) and (7) we deduce that $\mathcal{H}_\mu \subset L^1(0, \infty)$. Since $\mathcal{D}(0, \infty) \subset \mathcal{H}_\mu$ then \mathcal{H}_μ is dense in $L^1(0, \infty)$. Given $f \in L^1(0, \infty)$ and $\{\phi_n\} \subset \mathcal{H}_\mu$ such that $\phi_n \rightarrow f$ in $L^1(0, \infty)$ then by Remark 11 (see Appendix) $h_\mu(\phi_n) \rightarrow h_\mu(f)$ uniformly. Since $h_\mu(\phi_n) \in C_0(0, \infty)$ then $h_\mu(f) \in C_0(0, \infty)$. \square

Remark 12. *For $\mu > -\frac{1}{2}$, \mathcal{H}_μ is a dense subset of $L^2(0, \infty)$ and for $\phi \in \mathcal{H}_\mu$ we have that*

$$\|h_\mu \phi\|_2 = \|\phi\|_2,$$

So, we can consider the extension to $L^2(0, \infty)$ of h_μ and

$$\|h_\mu f\|_2 = \|f\|_2,$$

for $f \in L^2(0, \infty)$.

7.2. Hankel convolution.

Proposition 16. $D_\mu(x, y, z)$ satisfies the following properties:

- (1) $D_\mu(x, y, z) \geq 0$ for $x, y, z \in (0, \infty)$.
- (2) $\int_0^\infty \sqrt{zt} J_\mu(zt) D_\mu(x, y, z) dz = \sqrt{xt} J_\mu(xt) \sqrt{yt} J_\mu(yt) t^{-\mu-\frac{1}{2}}$ for $x, y, t \in (0, \infty)$.
- (3) $\int_0^\infty z^{\mu+\frac{1}{2}} D_\mu(x, y, z) dz = c_\mu^{-1} x^{\mu+\frac{1}{2}} y^{\mu+\frac{1}{2}}$ for $x, y \in (0, \infty)$.

Proof. Assertion (1) follows immediately.

To proof (2), first we observe that

$$\begin{aligned} \left| \sqrt{zt} J_\mu(zt) D_\mu(x, y, z) \right| &= \left| \sqrt{zt} (zt)^\mu (zt)^{-\mu} J_\mu(zt) D_\mu(x, y, z) \right| \leq C t^{\mu+\frac{1}{2}} \left| z^{\mu+\frac{1}{2}} D_\mu(x, y, z) \right| = \\ &= C t^{\mu+\frac{1}{2}} \left| z^{\mu+\frac{1}{2}} \frac{2^{\mu-1} (xyz)^{-\mu+\frac{1}{2}}}{\Gamma(\mu+\frac{1}{2}) \sqrt{\pi}} (A(x, y, z))^{2\mu-1} \right| = \\ &= C t^{\mu+\frac{1}{2}} \frac{(xy)^{-\mu+\frac{1}{2}}}{2^{3\mu-1} \Gamma(\mu+\frac{1}{2}) \sqrt{\pi}} \left| z((x+y)^2 - z^2)^{\mu-\frac{1}{2}} (z^2 - (x-y)^2)^{\mu-\frac{1}{2}} \right| \end{aligned}$$

and the last function is integrable for $z \in [|x-y|, x+y]$ and $\mu > -\frac{1}{2}$. So, we conclude that $z^{\mu+\frac{1}{2}} D_\mu(x, y, z)$ is integrable in $(0, \infty)$ for $\mu > -\frac{1}{2}$. Now, we consider the change of variables $T : (0, \pi) \rightarrow (0, \infty)$ given by $T(\phi) = \sqrt{x^2 + y^2 - 2xy \cos \phi}$. Then $|x-y| < T(\phi) < x+y$, $\frac{d}{d\phi} T(\phi) = \frac{xy \sin \phi}{\sqrt{x^2 + y^2 - 2xy \cos \phi}}$ and $A(x, y, T(\phi)) = \frac{xy}{2} \sin \phi$. So,

$$\begin{aligned} \int_0^\infty \sqrt{zt} J_\mu(zt) D_\mu(x, y, z) dz &= \frac{2^{\mu-1} (xy)^{-\mu+\frac{1}{2}}}{\Gamma(\mu+\frac{1}{2}) \sqrt{\pi}} \int_{|x-y|}^{x+y} \sqrt{zt} J_\mu(zt) z^{-\mu+\frac{1}{2}} (A(x, y, z))^{2\mu-1} dz = \\ &= \frac{2^{\mu-1} (xy)^{-\mu+\frac{1}{2}} t^{\frac{1}{2}}}{\Gamma(\mu+\frac{1}{2}) \sqrt{\pi}} \int_0^\pi J_\mu(\sqrt{x^2 + y^2 - 2xy \cos \phi} t) (\sqrt{x^2 + y^2 - 2xy \cos \phi})^{-\mu+1} \cdot \\ &\quad \cdot \left(\frac{xy}{2} \sin \phi \right)^{2\mu-1} \frac{xy \sin \phi}{\sqrt{x^2 + y^2 - 2xy \cos \phi}} d\phi = \\ (39) \quad &\frac{(xy)^{\mu+\frac{1}{2}} t^{\frac{1}{2}}}{2^\mu \Gamma(\mu+\frac{1}{2}) \sqrt{\pi}} \int_0^\pi \frac{J_\mu(\sqrt{x^2 + y^2 - 2xy \cos \phi} t)}{(\sqrt{x^2 + y^2 - 2xy \cos \phi})^\mu} \sin^{2\mu} \phi d\phi. \end{aligned}$$

Since

$$\int_0^\pi \frac{J_\mu(\sqrt{Z^2 + z^2 - 2zZ \cos \phi})}{(\sqrt{Z^2 + z^2 - 2zZ \cos \phi})^\mu} \sin^{2\mu} \phi d\phi = 2^\mu \Gamma\left(\mu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) \frac{J_\mu(Z)}{Z^\mu} \frac{J_\mu(z)}{z^\mu},$$

(see (16) pg. 367 [11]) valid to $z, Z > 0$, $\mu > -\frac{1}{2}$, and considering $Z = xt$ y $z = yt$ we obtain in the last equality

$$(40) \quad \int_0^\pi \frac{J_\mu(\sqrt{x^2 + y^2 - 2xy \cos \phi} t)}{(\sqrt{x^2 + y^2 - 2xy \cos \phi})^\mu t^\mu} \sin^{2\mu} \phi d\phi = 2^\mu \Gamma\left(\mu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) \frac{J_\mu(xt)}{(xt)^\mu} \frac{J_\mu(yt)}{(yt)^\mu}.$$

Applying (40) in (39), we obtain that

$$\begin{aligned} & \int_0^\infty \sqrt{zt} J_\mu(z) D_\mu(x, y, z) dz = \\ & \frac{(xy)^{\mu+\frac{1}{2}} t^{\frac{1}{2}}}{2^\mu \Gamma(\mu + \frac{1}{2}) \sqrt{\pi}} t^\mu 2^\mu \Gamma\left(\mu + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) \frac{J_\mu(xt)}{(xt)^\mu} \frac{J_\mu(yt)}{(yt)^\mu} = \sqrt{xt} J_\mu(xt) \sqrt{yt} J_\mu(yt) t^{-\mu-\frac{1}{2}}. \end{aligned}$$

As for (3), we consider again the change of variable T

$$\begin{aligned} & \int_0^\infty z^{\mu+\frac{1}{2}} D_\mu(x, y, z) dz = \frac{2^{\mu-1} (xy)^{-\mu+\frac{1}{2}}}{\Gamma(\mu + \frac{1}{2}) \sqrt{\pi}} \int_{|x-y|}^{x+y} z (A(x, y, z))^{2\mu-1} dz = \\ & \frac{2^{\mu-1} (xy)^{-\mu+\frac{1}{2}}}{\Gamma(\mu + \frac{1}{2}) \sqrt{\pi}} \int_0^\pi \sqrt{x^2 + y^2 - 2xy \cos \phi} \left(\frac{xy}{2} \sin \phi\right)^{2\mu-1} \frac{xy \sin \phi}{\sqrt{x^2 + y^2 - 2xy \cos \phi}} d\phi = \\ & = \frac{2^{\mu-1} (xy)^{-\mu+\frac{1}{2}}}{\Gamma(\mu + \frac{1}{2}) \sqrt{\pi}} 2^{-2\mu+1} (xy)^{2\mu} \frac{\sqrt{\pi} \Gamma(\mu + \frac{1}{2})}{\mu \Gamma(\mu)} = \\ & = (xy)^{\mu+\frac{1}{2}} (2^\mu \Gamma(\mu + 1))^{-1}. \end{aligned}$$

□

Proof of Theorem 1

Proof. (1) Let $f \in L^1(sr)$ and $g \in L^\infty(r)$, then:

$$\begin{aligned} & \int_0^\infty |f(y)| \left[\int_0^\infty |g(z)| D_\mu(x, y, z) dz \right] dy \leq \\ & \leq \|g\|_{L^\infty(r)} \int_0^\infty |f(y)| \left[\int_0^\infty z^{\mu+\frac{1}{2}} D_\mu(x, y, z) dz \right] dy = \\ (41) \quad & \|g\|_{L^\infty(r)} \int_0^\infty |f(y)| y^{\mu+\frac{1}{2}} x^{\mu+\frac{1}{2}} c_\mu^{-1} dy = x^{\mu+\frac{1}{2}} \|f\|_{L^1(sr)} \|g\|_{L^\infty(r)}, \end{aligned}$$

thus (16) exists for every $x \in (0, \infty)$ and by (41) we have

$$|x^{-\mu-\frac{1}{2}}(f \sharp g)| \leq \|f\|_{L^1(sr)} \|g\|_{L^\infty(r)},$$

hence (18).

(2) Given $f \in L^1(sr)$ and $g \in L^p(sr^p)$ with $1 \leq p < \infty$, set:

$$(42) \quad K(x, z) = \int_0^\infty x^{-\mu-\frac{1}{2}} z^{-\mu-\frac{1}{2}} f(y) D_\mu(x, y, z) c_\mu dy.$$

We claim that:

$$(1)' \int_0^\infty |K(x, z)| s(x) dx \leq \|f\|_{L^1(sr)} = \|rf\|_{L^1(s)};$$

$$(2)' \int_0^\infty |K(x, z)| s(z) dz \leq \|f\|_{L^1(sr)} = \|rf\|_{L^1(s)}.$$

In fact

$$\begin{aligned} \int_0^\infty |K(x, z)| s(x) dx &= \int_0^\infty |K(x, z)| x^{2\mu+1} c_\mu^{-1} dx = \\ \int_0^\infty \left| \int_0^\infty x^{-\mu-\frac{1}{2}} z^{-\mu-\frac{1}{2}} f(y) D_\mu(x, y, z) c_\mu dy \right| x^{2\mu+1} c_\mu^{-1} dx &\leq \\ \int_0^\infty \left[\int_0^\infty x^{\mu+\frac{1}{2}} D_\mu(x, y, z) dx \right] |f(y)| z^{-\mu-\frac{1}{2}} dy &= \\ \int_0^\infty c_\mu^{-1} y^{\mu+\frac{1}{2}} z^{\mu+\frac{1}{2}} |f(y)| z^{-\mu-\frac{1}{2}} dy &= \|f\|_{L^1(sr)} \end{aligned}$$

The proof for (2)' is similar. If $h \in L^p(s)$ then the integral

$$Th(x) = \int_0^\infty K(x, z) h(z) s(z) dz.$$

converges absolutely for a.e. $x \in (0, \infty)$ (see [2], theorem 6.18), also $Th \in L^p(s)$ and

$$(43) \quad \|Th\|_{L^p(s)} \leq \|f\|_{L^1(rs)} \|h\|_{L^p(s)}.$$

Then, since $g \in L^p(sr^p)$ then $h = rg \in L^p(s)$ and we have that

$$\begin{aligned} T(rg)(x) &= \int_0^\infty K(x, z) z^{-\mu-\frac{1}{2}} g(z) c_\mu^{-1} z^{2\mu+1} dz = \\ \int_0^\infty \left[\int_0^\infty x^{-\mu-\frac{1}{2}} z^{-\mu-\frac{1}{2}} f(y) D_\mu(x, y, z) c_\mu dy \right] z^{-\mu-\frac{1}{2}} g(z) c_\mu^{-1} z^{2\mu+1} dz &= \\ (44) \quad x^{-\mu-\frac{1}{2}} \int_0^\infty \left[\int_0^\infty f(y) D_\mu(x, y, z) dy \right] g(z) dz & \end{aligned}$$

With similar considerations applied to $|f| \in L^1(sr)$ and $|g| \in L^p(sr^p)$ we obtain that for a.e. $x \in (0, \infty)$ the integral

$$\int_0^\infty \left[\int_0^\infty |f(y)| D_\mu(x, y, z) dy \right] |g(z)| dz$$

is finite and by application of Tonelli-Hobson theorem in (44) we conclude that

$$T(rg)(x) = x^{-\mu-\frac{1}{2}}(f \sharp g)(x).$$

From the previous equality and (43) we have

$$\|r(f \sharp g)\|_{L^p(s)} \leq \|f\|_{L^1(rs)} \|rg\|_{L^p(s)},$$

and so (19) is valid. \square

Proof of Theorem 2

Proof. By (1) and (3) of Proposition 16 we have that

$$\int_0^\infty \int_0^\infty x_0^{-\mu-\frac{1}{2}} y^{\mu+\frac{1}{2}} D_\mu(x_0, y, z) \phi_n(z) dy dz = 1,$$

then

$$f \sharp \phi_n(x_0) - f(x_0) = \int_0^\infty \int_0^\infty D_\mu(x_0, y, z) \phi_n(z) y^{\mu+\frac{1}{2}} (y^{-\mu-\frac{1}{2}} f(y) - x_0^{-\mu-\frac{1}{2}} f(x_0)) dy dz.$$

By continuity of f in x_0 let $\delta > 0$ such that $|y^{-\mu-\frac{1}{2}} f(y) - x_0^{-\mu-\frac{1}{2}} f(x_0)| < \varepsilon$ if $|y - x_0| < \delta$, and we consider

$$|f \sharp \phi_n(x_0) - f(x_0)| \leq |I_1| + |I_2|,$$

where

(45)

$$|I_1| = \left| \int_0^\delta \int_0^\infty D_\mu(x_0, y, z) \phi_n(z) y^{\mu+\frac{1}{2}} (y^{-\mu-\frac{1}{2}} f(y) - x_0^{-\mu-\frac{1}{2}} f(x_0)) dy dz \right|$$

(46)

$$|I_2| = \left| \int_\delta^\infty \int_0^\infty D_\mu(x_0, y, z) \phi_n(z) y^{\mu+\frac{1}{2}} (y^{-\mu-\frac{1}{2}} f(y) - x_0^{-\mu-\frac{1}{2}} f(x_0)) dy dz \right|$$

Since $D_\mu(x_0, y, z) \neq 0$ only if $|x_0 - z| < y < x_0 + z$, and if $0 < z < \delta$ then $(|x_0 - z|, x_0 + z) \subset (x_0 - \delta, x_0 + \delta)$, then we obtain in (45) that

$$|I_1| \leq \varepsilon \int_0^\delta \int_0^\infty D_\mu(x_0, y, z) \phi_n(z) y^{\mu+\frac{1}{2}} dy dz =$$

$$\varepsilon \int_0^\delta x_0^{\mu+\frac{1}{2}} z^{\mu+\frac{1}{2}} c_\mu^{-1} \phi_n(z) dz \leq \varepsilon x_0^{\mu+\frac{1}{2}}.$$

On the other hand

$$\begin{aligned} |I_2| &\leq 2\|f\|_{L^\infty(r)} \int_\delta^\infty \int_0^\infty D_\mu(x_0, y, z) \phi_n(z) y^{\mu+\frac{1}{2}} dy dz = \\ &2\|f\|_{L^\infty(r)} x_0^{\mu+\frac{1}{2}} \int_\delta^\infty c_\mu^{-1} z^{\mu+\frac{1}{2}} \phi_n(z) dz \end{aligned}$$

so, $|I_2| \rightarrow 0$ when $n \rightarrow \infty$ and the first assertion has been proven. Second affirmation follows from the previous proof and the uniformly continuity of rf . \square

Proof of Proposition 5

Proof. If $f, g \in L^1(sr)$, then

$$\begin{aligned} h_\mu(f \sharp g)(t) &= \int_0^\infty f \sharp g(x) \sqrt{xt} J_\mu(xt) dx = \\ (47) \quad &\int_0^\infty \left[\int_0^\infty \int_0^\infty D_\mu(x, y, z) f(y) g(z) dy dz \right] \sqrt{xt} J_\mu(xt) dx \end{aligned}$$

Since $f \sharp g \in L^1(sr)$ we obtain that

$$\int_0^\infty \left[\int_0^\infty \int_0^\infty |D_\mu(x, y, z) f(y) g(z)| dy dz \right] |\sqrt{xt} J_\mu(xt)| dx < \infty$$

for all $t \in (0, \infty)$. Then,

$$\begin{aligned} h_\mu(f \sharp g)(t) &= \int_0^\infty \int_0^\infty f(y) g(z) \left[\int_0^\infty \sqrt{xt} J_\mu(xt) D_\mu(x, y, z) dx \right] dy dz = \\ &\int_0^\infty \int_0^\infty f(y) g(z) \sqrt{yt} J_\mu(yt) \sqrt{zt} J_\mu(zt) t^{-\mu-\frac{1}{2}} dy dz = t^{-\mu-\frac{1}{2}} h_\mu(f)(t) h_\mu(g)(t). \end{aligned}$$

\square

7.3. Properties of N_λ . Proof of Lemma 1

Proof. a)

$$\begin{aligned} \|N_\lambda\|_{L^1(sr)} &= \frac{1}{c_\mu} \int_0^\infty \lambda^{\frac{\mu}{2}} x^{\frac{1}{2}} \mathcal{K}_\mu(\sqrt{\lambda} x) x^{\mu+\frac{1}{2}} dx = \\ &= \frac{1}{c_\mu} \int_0^\infty \lambda^{\frac{\mu}{2}} x^{\frac{1}{2}} \frac{1}{2} \left(\frac{1}{2} \sqrt{\lambda} x \right)^\mu \left[\int_0^\infty e^{-t-\frac{\lambda x^2}{4t}} \frac{dt}{t^{\mu+1}} \right] x^{\mu+\frac{1}{2}} dx = \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{c_\mu} \left(\frac{1}{2}\right)^{\mu+1} \lambda^\mu \int_0^\infty \left[\int_0^\infty x^{2\mu+1} e^{-\frac{\lambda x^2}{4t}} dx \right] e^{-t} \frac{dt}{t^{\mu+1}} = \\
&= \frac{1}{c_\mu} 2^\mu \lambda^{-1} \Gamma(\mu+1) \int_0^\infty e^{-t} dt = \lambda^{-1}.
\end{aligned}$$

For b), in virtue of the following equality

$$(48) \quad \int_0^\infty x^{\mu+\frac{1}{2}} e^{-\frac{x^2}{2}} \sqrt{xy} J_\mu(xy) dx = y^{\mu+\frac{1}{2}} e^{-\frac{y^2}{2}}, \quad y > 0,$$

(see [10, (5.9), pp. 46]), setting $y = (\sqrt{a})^{-1}r$ with $a, r > 0$, and considering the change of variable $s = \frac{x}{\sqrt{a}}$, then we obtain that

$$\int_0^\infty (\sqrt{a} s)^{\mu+\frac{1}{2}} e^{-\frac{as^2}{2}} \sqrt{sr} J_\mu(sr) \sqrt{a} ds = \left(\frac{r}{\sqrt{a}}\right)^{\mu+\frac{1}{2}} e^{-\frac{r^2}{2a}}$$

so,

$$(49) \quad \int_0^\infty s^{\mu+1} e^{-\frac{as^2}{2}} J_\mu(sr) ds = a^{-\mu-1} r^\mu e^{-\frac{r^2}{2a}},$$

for all $a > 0$. Then,

$$\begin{aligned}
h_\mu N_\lambda(y) &= \int_0^\infty \lambda^{\frac{\mu}{2}} x^{\frac{1}{2}} \mathcal{K}_\mu(\sqrt{\lambda} x) \sqrt{xy} J_\mu(xy) dx = \\
&= \int_0^\infty \lambda^{\frac{\mu}{2}} x^{\frac{1}{2}} \frac{1}{2} \left(\frac{1}{2} \sqrt{\lambda} x\right)^\mu \left[\int_0^\infty e^{-t-\frac{\lambda x^2}{4t}} \frac{dt}{t^{\mu+1}} \right] \sqrt{xy} J_\mu(xy) dx = \\
(50) \quad &\lambda^\mu y^{\frac{1}{2}} \left(\frac{1}{2}\right)^{\mu+1} \int_0^\infty x^{\mu+1} \left[\int_0^\infty e^{-t-\frac{\lambda x^2}{4t}} \frac{dt}{t^{\mu+1}} \right] J_\mu(xy) dx.
\end{aligned}$$

Applying the boundedness of function $z^{-\mu} J_\mu(z)$ we obtain that

$$\begin{aligned}
&\int_0^\infty x^{\mu+1} \left[\int_0^\infty e^{-t-\frac{\lambda x^2}{4t}} \frac{dt}{t^{\mu+1}} \right] |J_\mu(xy)| dx = \\
&y^\mu \int_0^\infty \left[\int_0^\infty e^{-\frac{\lambda x^2}{4t}} x^{2\mu+1} |(xy)^{-\mu} J_\mu(xy)| dx \right] e^{-t} \frac{dt}{t^{\mu+1}} < \infty
\end{aligned}$$

Then, reversing the order of integration in (50) and applying (49) we obtain that

$$\begin{aligned}
h_\mu N_\lambda(y) &= \lambda^\mu y^{\frac{1}{2}} \left(\frac{1}{2}\right)^{\mu+1} \int_0^\infty \left[\int_0^\infty x^{\mu+1} e^{-\left(\frac{\lambda}{2t}\right)\frac{x^2}{2}} J_\mu(xy) dx \right] e^{-t} \frac{dt}{t^{\mu+1}} = \\
&\lambda^\mu y^{\frac{1}{2}} \left(\frac{1}{2}\right)^{\mu+1} \int_0^\infty \left(\frac{\lambda}{2t}\right)^{-\mu-1} y^\mu e^{-\frac{ty^2}{\lambda}} e^{-t} \frac{dt}{t^{\mu+1}} = \lambda^{-1} y^{\mu+\frac{1}{2}} \int_0^\infty e^{-t(1+\frac{y^2}{\lambda})} dt.
\end{aligned}$$

Considering the change of variable $s = t(1 + \frac{y^2}{\lambda})$ in the last integral we obtain finally

$$h_\mu N_\lambda(y) = \lambda^{-1} y^{\mu+\frac{1}{2}} \int_0^\infty e^{-s} \left(1 + \frac{y^2}{\lambda}\right)^{-1} ds = \lambda^{-1} y^{\mu+\frac{1}{2}} \left(1 + \frac{y^2}{\lambda}\right)^{-1} = \frac{y^{\mu+\frac{1}{2}}}{\lambda + y^2}.$$

□

Proof of Lemma 2.

Proof. Suppose that $f \in L^p(sr^p)$ and $\psi \in \mathcal{H}_\mu$, we claim that

$$(51) \quad \int_0^\infty (N_\lambda \sharp f)(x) \psi(x) dx = \int_0^\infty f(z) (N_\lambda \sharp \psi)(z) dz.$$

Indeed, we first observe that the following integral is finite

$$\int_0^\infty |f(z)| \left[\int_0^\infty \int_0^\infty |N_\lambda(y)| |\psi(x)| D_\mu(x, y, z) dx dy \right] dz$$

In fact, given a integer q such that $\frac{1}{p} + \frac{1}{q} = 1$, the function

$$G(z) = \int_0^\infty \int_0^\infty |N_\lambda(y)| |\psi(x)| D_\mu(x, y, z) dx dy$$

is in $L^q(sr^q)$ because it is the convolution of $|N_\lambda(y)| \in L^1(sr)$ and $|\psi(x)| \in L^q(sr^q)$ ($\psi \in \mathcal{H}_\mu$). Since $f \in L^p(sr^p)$ then

$$\int_0^\infty |f(z)| G(z) dz = \int_0^\infty (r |f(z)|)(s^{-1} r^{-1} G(z)) s dz < \infty$$

because $r |f| \in L^p(s)$ and $s^{-1} r^{-1} G = c_\mu r G \in L^q(s)$. Then

$$\begin{aligned} \int_0^\infty f(z) (N_\lambda \sharp \psi)(z) dz &= \int_0^\infty f(z) \left[\int_0^\infty \int_0^\infty N_\lambda(y) \psi(x) D_\mu(x, y, z) dx dy \right] dz \\ &= \int_0^\infty \left[\int_0^\infty \int_0^\infty f(z) N_\lambda(y) D_\mu(x, y, z) dz dy \right] \psi(x) dx \\ &= \int_0^\infty (N_\lambda \sharp f)(x) \psi(x) dx \end{aligned}$$

and we thus get (51). The proof for $f \in L^\infty(r)$ is similar.

Now, given $\phi \in \mathcal{H}_\mu$ and $f \in L^p(sr^p)$ or $L^\infty(r)$, by (51), we have that

$$(52) \quad (h_\mu(N_\lambda \sharp f), \phi) = ((N_\lambda \sharp f), h_\mu \phi) = \int_0^\infty f(x) N_\lambda \sharp h_\mu \phi(x) dx$$

By Propositions 3, 14 and item b) of Lemma 1

$$h_\mu(N_\lambda \sharp h_\mu \phi)(y) = r h_\mu(N_\lambda) h_\mu(h_\mu \phi)(y) = \frac{\phi(y)}{\lambda + y^2}.$$

So,

$$(53) \quad N_\lambda \# h_\mu \phi = h_\mu \left(\frac{\phi}{\lambda + y^2} \right).$$

Finally, from (52) and (53) we obtain that for $\phi \in \mathcal{H}_\mu$ that

$$\begin{aligned} (h_\mu(N_\lambda \# f), \phi) &= \int_0^\infty f(x) N_\lambda \# h_\mu \phi(x) dx = \int_0^\infty f(x) h_\mu \left(\frac{\phi}{\lambda + y^2} \right)(x) dx = \\ &= \int_0^\infty \frac{1}{\lambda + x^2} h_\mu(f)(x) \phi(x) dx = \left(\frac{1}{\lambda + x^2} h_\mu(f), \phi \right) \end{aligned}$$

□

Acknowledgments Some of the main ideas of this paper were discussed with Miguel Sanz. The author wishes to thank him for the many helpful suggestions and for the stimulating conversations.

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